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Relating harmonic and projective descriptions of $\mathcal{N} = 2$ nonlinear sigma models

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ABSTRACT: Recent papers have established the relationship between projective superspace and a complexified version of harmonic superspace. We extend this construction to the case of general nonlinear sigma models in both frameworks. Using an analogy with Hamiltonian mechanics, we demonstrate how the Hamiltonian structure of the harmonic action and the symplectic structure of the projective action naturally arise from a single unifying action on a complexified version of harmonic superspace. This links the harmonic and projective descriptions of hyperkähler target spaces. For the two examples of Taub-NUT and Eguchi-Hanson, we show how to derive the projective superspace solutions from the harmonic superspace solutions.

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1 Introduction and review

The relation between supersymmetry and complex geometry began with Zumino’s observation that $\mathcal{N} = 1$ supersymmetric nonlinear sigma models must possess a Kähler geometry [1]. Soon after, it was discovered that $\mathcal{N} = 2$ supersymmetry requires that the geometry be hyperkähler [2] (see also [3]). However, while $\mathcal{N} = 1$ superspace naturally leads to a real Kähler potential as the most general Lagrangian, the link between $\mathcal{N} = 2$ superspace and hyperkähler geometry is more complicated, due mainly to the complexity of the superspaces that furnish off-shell representations of $\mathcal{N} = 2$ supersymmetry.

Harmonic superspace [4, 5] extends the usual flat $\mathcal{N} = 2$ superspace by an S^2 and deals with fields and operators which are globally defined on this auxiliary manifold. Projective superspace [6–8] uses the same auxiliary manifold, though usually interpreted as \mathbb{CP}^1 , but requires fields and operators to be holomorphic in the \mathbb{CP}^1 coordinate. The most general fields and actions in projective superspace are necessarily singular somewhere on the auxiliary manifold.

In 1998, Kuzenko showed that projective superspace could be understood as a double-punctured harmonic superspace [9]. The main idea was to embed projective multiplets into globally defined harmonic multiplets that were holomorphic everywhere except a small region around the poles of the S^2 (using the equation of motion for the hypermultiplet and a gauge condition for the vector multiplet). More recently, Jain and Siegel [10] (see also [11, 12]) have constructed projective fields and actions from harmonic ones using a different approach. Their idea is to complexify the S^2 and to interpret one of its dimensions as an

additional coordinate which can be integrated over once the equation of motion (for a hypermultiplet) or a gauge condition (for a vector multiplet) is imposed.

Our goal in this paper is to take the same basic idea as [10] and apply it to the most general nonlinear sigma model constructed from hypermultiplets in harmonic superspace. We will show that there exists a natural identification with a model in projective superspace. Moreover, once the nonlinear sigma model equations of motion are solved in harmonic superspace, the corresponding projective superspace action and solution can be derived. We will demonstrate the procedure for three examples: the free hypermultiplet, Taub-NUT, and Eguchi-Hanson.

To keep this paper as self-contained as possible, we end this section with a brief review of harmonic and projective superspaces and how both describe nonlinear sigma models. These briefest of sketches are meant only to remind the reader of the details and to clarify our conventions. We also include a very brief review of an alternative action formulation of Hamiltonian mechanics that will be necessary for our construction.

1.1 Nonlinear sigma models in harmonic superspace

Harmonic superspace [4, 5] involves the extension of the global $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$, parametrized by $z^M = (x^m, \theta_i^\alpha, \bar{\theta}_\alpha^i)$, with the two-sphere $S^2 \cong SU(2)/U(1)$, parametrized by the harmonics u_i^\pm . The harmonics obey the relations

$$\overline{u^{i+}} = u_i^- , \quad u^{i+} u_i^- = 1 \quad (1.1)$$

and describe the space $SU(2)$. Associated with the harmonics are three derivative operations

$$D^{++} = u_i^+ \frac{\partial}{\partial u_i^-} , \quad D^{--} = u_i^- \frac{\partial}{\partial u_i^+} , \quad D^0 = u^{i+} \frac{\partial}{\partial u^{i+}} - u_i^- \frac{\partial}{\partial u_i^-} , \quad (1.2)$$

corresponding to the three generators of $SU(2)$. All superfields and operators in harmonic superspace are required to possess a fixed $U(1)$ charge, that is, to be eigenstates of the operator D^0 . This allows the harmonics to be identified modulo an overall phase,

$$u_i^\pm \sim e^{\pm i\varphi} u_i^\pm , \quad \varphi \in \mathbb{R} , \quad (1.3)$$

which ensures that only the coset space $SU(2)/U(1) \cong S^2$ is being described.¹

To construct an $\mathcal{N} = 2$ supersymmetric action, one needs a Lagrangian \mathcal{L}^{+4} of charge +4, which is globally defined on S^2 and obeys the analyticity conditions

$$D_\alpha^+ \mathcal{L}^{+4} = \bar{D}_{\dot{\alpha}}^+ \mathcal{L}^{+4} = 0 , \quad D_\alpha^+ := u_i^+ D_\alpha^i , \quad \bar{D}_{\dot{\alpha}}^+ := u_i^+ \bar{D}_{\dot{\alpha}}^i . \quad (1.4)$$

This ensures that \mathcal{L}^{+4} depends on only half the Grassmann coordinates:

$$\begin{aligned} \mathcal{L}^{+4} &= \mathcal{L}^{+4}(x_A, \theta^+, \bar{\theta}^+, u^\pm) , \quad \theta^\pm := \theta^i u_i^\pm , \quad \bar{\theta}^\pm := \bar{\theta}^i u_i^\pm , \\ x_A^m &:= x^m - i\theta^+ \sigma^m \bar{\theta}^- - i\theta^- \sigma^m \bar{\theta}^+ . \end{aligned}$$

¹To prove $SU(2)/U(1) \cong S^2$, introduce the coordinate $X^I \in \mathbb{R}^3$ given by $X^I = u_j^- (\sigma^I)^j_k u^{k+}$ where σ^I are the three Pauli matrices. One can check that $\sum_I (X^I)^2 = 1$. Given any such X^I , one can reconstruct the harmonics up to the equivalence relation (1.3).

In analogy to $\mathcal{N} = 1$ chiral actions, we can introduce an action as an integral over half the Grassmann coordinates,²

$$S = \int du \int d^4x d^4\theta^+ \mathcal{L}^{+4} = \frac{1}{16} \int d^4x \int du (D^-)^2 (\bar{D}^-)^2 \mathcal{L}^{+4} \quad (1.5)$$

where

$$D_\alpha^- := u_i^- D_\alpha^i, \quad \bar{D}_{\dot{\alpha}}^- := u_i^- \bar{D}_{\dot{\alpha}}^i, \quad (1.6)$$

and $\int du$ denotes the integration over S^2 .

To ensure that the action is real, we require the Lagrangian to be real under a generalized complex conjugation $\widetilde{}$, called smile conjugation. It corresponds to normal complex conjugation followed by the antipodal map. The smile conjugates of some relevant quantities are

$$\widetilde{u^{i\pm}} = -u_i^\pm, \quad \widetilde{u_i^\pm} = +u^{i\pm}, \quad \widetilde{D^{++}} = D^{++}. \quad (1.7)$$

Within harmonic superspace, the hypermultiplet is described by a complex analytic superfield q^+ possessing an infinite set of auxiliary fields,

$$q^+ = q^i u_i^+ + q^{(ijk)} u_i^+ u_j^+ u_k^- + \dots \quad (1.8)$$

The free hypermultiplet possesses the equation of motion $D^{++} q^+ = 0$, which eliminates all but the first term in this expansion, and so on-shell q^+ is described by a superfield q^i obeying $D_\alpha^{(i} q^{j)} = \bar{D}_{\dot{\alpha}}^{(i} q^{j)} = 0$. More generally, the Lagrangian involving several hypermultiplets q^{a+} is given by [13, 14] (see also [15])

$$\mathcal{L}^{+4} = \frac{1}{2} q_a^+ D^{++} q^{a+} + H^{+4}, \quad H^{+4} = H^{+4}(q^{a+}, u_i^\pm), \quad (1.9)$$

where $q_a^+ = \Omega_{ab} q^{b+}$ for the canonical symplectic form Ω_{ab} .³ The analytic function H^{+4} is a function of q^{a+} and u_i^\pm of $U(1)$ charge $+4$. In order for the action to be real, we require that

$$\widetilde{H^{+4}} = H^{+4}, \quad \widetilde{q^{a+}} = -q_a^+, \quad \widetilde{q_a^+} = q^{a+}. \quad (1.10)$$

We can choose a basis

$$q^{a+} = (q^{I+}, p_I^+), \quad q_a^+ = (-p_I^+, q^{I+}), \quad I = 1, \dots, n \quad (1.11)$$

with $\widetilde{q^{I+}} = p_I^+$ and $\widetilde{p_I^+} = -q^{I+}$. The action becomes in this basis

$$S = \int du \int d^4x d^4\theta^+ \left(-p_I^+ D^{++} q^{I+} + H^{+4} \right). \quad (1.12)$$

²More precisely, the action involves an integral over the analytic coordinate x_A rather than x . But as with $\mathcal{N} = 1$ chiral actions, the difference amounts to a total derivative.

³A more general action is possible, but it can always be put into this form by a certain gauge transformation.

The equations of motion are

$$D^{++}q^{I+} = \frac{\partial H^{+4}}{\partial p_I^+}, \quad D^{++}p_I^+ = -\frac{\partial H^{+4}}{\partial q^{I+}}. \quad (1.13)$$

It was further shown in [14] that *any* $4n$ -dimensional hyperkähler target space geometry can be derived (at least locally) from some harmonic superspace system of this form. The target space geometry is encoded in the $\theta = 0$ parts of q^{I+} and p_I^+ , denoted $q^{I+}|$ and $p_I^+|$, which turn out to be functions of the $2n$ complex fields $q^{Ii}(x)$ and $p_I^i(x)$. In particular, the three Kähler two-forms Ω^{ij} describing the hyperkähler geometry can be encoded in a single two-form⁴

$$\Omega^{++} = u_i^+ u_j^+ \Omega^{ij} = (dq^{I+} \wedge dp_I^+)|, \quad (1.14)$$

carrying $U(1)$ charge $+2$. Remarkably, this quantity is formally independent of u_i^- .

It was noted in [16] that the equations (1.13) bear a strong resemblance to Hamilton's equations for a symplectic system. Similarly, (1.14) resembles the canonical two-form. This observation is one of the two ingredients of our construction.

Finally, we should mention that although a specific coordinate realization of the harmonics is usually unnecessary, we will find it useful to introduce a particular parametrization [17] in terms of an isotwistor v^i and its conjugate \bar{v}_i ,

$$u^{i+} = \frac{v^i}{\sqrt{v^k \bar{v}_k}}, \quad u_i^- = \frac{\bar{v}_i}{\sqrt{v^k \bar{v}_k}}. \quad (1.15)$$

These relations fix v^i only up to a real factor. Together with the identification (1.3), the coordinate v^i is defined modulo the equivalence relation

$$v^i \sim c v^i, \quad c \in \mathbb{C} - \{0\}. \quad (1.16)$$

The isotwistor v^i is the homogeneous coordinate for the space \mathbb{CP}^1 and this parametrization realizes the equivalence $\mathbb{CP}^1 \cong S^2$. Under smile conjugation, $\tilde{v}^i = -\varepsilon_{ij} v^j$ and $\tilde{\bar{v}}_i = \varepsilon^{ij} \bar{v}_j$.

In the northern chart of \mathbb{CP}^1 where $v^\perp \neq 0$, one may introduce the inhomogeneous coordinate $\zeta = v^2/v^\perp$. Then $v^i = v^\perp(1, \zeta)$ and the harmonics can be written [5]

$$u^{i+} = \frac{e^{i\psi}}{\sqrt{1 + \zeta \bar{\zeta}}}(1, \zeta), \quad u_i^- = \frac{e^{-i\psi}}{\sqrt{1 + \zeta \bar{\zeta}}}(1, \bar{\zeta}), \quad e^{i\psi} = \frac{v^\perp}{\sqrt{v^\perp \bar{v}_\perp}}. \quad (1.17)$$

The real coordinate ψ parametrizes the $U(1)$ in the coset. The complex coordinate ζ is the usual coordinate describing the Riemann sphere, arising from stereographic projection from the south pole; the north pole corresponds to the point $\zeta = 0$ and the south pole to $\zeta = \infty$. In these sets of coordinates, the integral over S^2 is given by [5, 17]

$$\int du = \frac{1}{2\pi i} \int \frac{v^i dv_i \wedge \bar{v}_j d\bar{v}^j}{(v^k \bar{v}_k)^2} = \frac{i}{2\pi} \int \frac{d\zeta \wedge d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2}. \quad (1.18)$$

⁴An equivalent form of this expression appeared in eq. (A.9) of [14]. Using their eq. (A.6) and making the canonical choice for the symplectic matrix yields our eq. (1.14).

1.2 Nonlinear sigma models in projective superspace

Projective superspace [6–8] extends the $\mathcal{N} = 2$ supermanifold $\mathbb{R}^{4|8}$ by the real projective space \mathbb{CP}^1 parametrized by the complex isotwistor coordinate v^i defined up to the equivalence (1.16). Because $\mathbb{CP}^1 \cong S^2$, it is possible to describe quantities in projective superspace using the language established for harmonic.⁵

Let us begin with the action principle. A projective Lagrangian \mathcal{L}^{++} is a function of $U(1)$ charge +2 which is analytic,

$$D_\alpha^+ \mathcal{L}^{++} = \bar{D}_{\dot{\alpha}}^+ \mathcal{L}^{++} = 0 , \quad (1.19)$$

real under smile conjugation, $\widetilde{\mathcal{L}^{++}} = \mathcal{L}^{++}$, and constrained to obey

$$D^{++} \mathcal{L}^{++} = 0 \quad (1.20)$$

in some region on the S^2 . This implies that the function $\mathcal{L}^{(2)} := (v^k \bar{v}_k) \mathcal{L}^{++}$ is holomorphic in v^i of degree two,

$$\frac{\partial \mathcal{L}^{(2)}}{\partial \bar{v}_i} = 0 , \quad \mathcal{L}^{(2)}(cv) = c^2 \mathcal{L}^{(2)}(v) . \quad (1.21)$$

In general, \mathcal{L}^{++} cannot be globally defined on S^2 . This is the major difference with the harmonic action principle: we exchange global definition on S^2 for holomorphy on \mathbb{CP}^1 .

Nevertheless, we can construct an action principle if there exists some contour C in S^2 along which \mathcal{L}^{++} is well-defined:

$$S = \frac{1}{2\pi} \oint_C u^+ du^+ \int d^4x (D^-)^4 \mathcal{L}^{++} , \quad u^+ du^+ := u^{i+} du_i^+ . \quad (1.22)$$

The contour integral is naturally understood as

$$\frac{1}{2\pi} \oint_C u^+ du^+ = \frac{1}{2\pi} \oint_C \frac{v^i dv_i}{(v^k \bar{v}_k)} = -\frac{1}{2\pi} \oint_C \frac{e^{2i\psi} d\zeta}{1 + \zeta \bar{\zeta}} . \quad (1.23)$$

As in the harmonic case, we may reformulate the action principle to involve Grassmann integration,

$$S = \frac{1}{2\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ \mathcal{L}^{++} . \quad (1.24)$$

General projective hypermultiplets are naturally described by complex superfields Υ^+ [7, 20] which obey

$$D_\alpha^+ \Upsilon^+ = \bar{D}_{\dot{\alpha}}^+ \Upsilon^+ = 0 , \quad D^{++} \Upsilon^+ = 0 . \quad (1.25)$$

Since Υ^+ cannot be globally defined, let us restrict to superfields defined near the north pole, $\zeta = 0$. They are given by a Taylor series

$$\Upsilon^+ = u^{1+} \sum_{n=0}^{\infty} \Upsilon_n \zeta^n . \quad (1.26)$$

⁵In fact, both approaches can be understood as special cases of an isotwistor approach pioneered by Rosly [17, 18]. See [19] for a review.

Such superfields are called “arctic” [20]. Their smile conjugates $\check{\Upsilon}^+ = \widetilde{\Upsilon}^+$ are antarctic superfields,

$$\check{\Upsilon}^+ = u^{2+} \sum_{n=0}^{\infty} \bar{\Upsilon}_n \left(-\frac{1}{\zeta} \right)^n, \quad (1.27)$$

and are regular at $\zeta = \infty$ in the southern chart of S^2 .

Nonlinear sigma models are described by projective Lagrangians⁶

$$\mathcal{L}^{++} = F^{++}(\Upsilon^I, \check{\Upsilon}_I^+, u^{i+}), \quad I = 1, \dots, n \quad (1.28)$$

where F^{++} is real under smile-conjugation and possessing $U(1)$ charge +2. By construction, it is analytic and annihilated by D^{++} . Introducing the functions Υ , $\check{\Upsilon}$, and F given by⁷

$$\Upsilon^{I+} = u^{1+} \Upsilon^I, \quad \check{\Upsilon}_I^+ = u^{2+} \check{\Upsilon}_I, \quad F^{++} = i u^{1+} u^{2+} F(\Upsilon^I, \check{\Upsilon}_I, \zeta), \quad (1.29)$$

the nonlinear sigma model action can be written as an integral over the $\mathcal{N} = 1$ superspace

$$S = \int d^4x d^2\theta_{\perp} d^2\bar{\theta}^{\perp} L(\Upsilon_n^I, \bar{\Upsilon}_{In}), \quad L = \oint \frac{d\zeta}{2\pi i \zeta} F(\Upsilon^I, \check{\Upsilon}_I, \zeta). \quad (1.30)$$

The arctic superfield Υ^I is made of an infinite number of $\mathcal{N} = 1$ superfields Υ_n^I , the lowest two of which, $\Phi^I := \Upsilon_0^I$ and $\Sigma^I := \Upsilon_1^I$, are constrained,

$$\bar{D}_{\perp}^{\dot{\alpha}} \Phi^I = 0, \quad (\bar{D}_{\perp})^2 \Sigma^I = 0. \quad (1.31)$$

One can eliminate Υ_n^I for $n \geq 2$ by solving (formally in general but explicitly for a broad class of models [21, 22, 26–29]) the algebraic equations [21–23]

$$0 = \frac{\partial L}{\partial \Upsilon_n^I} = \oint \frac{d\zeta}{2\pi i \zeta} \frac{\partial F}{\partial \Upsilon^I} \zeta^n, \quad n \geq 2 \quad (1.32)$$

and similarly for $\bar{\Upsilon}_{In}$. The resulting action depends solely on Φ^I and Σ^I . The final step is to perform a duality transformation exchanging the complex linear superfield Σ^I for a chiral superfield Ψ_I , yielding an $\mathcal{N} = 1$ nonlinear sigma model action

$$S = \int d^4x d^2\theta_{\perp} d^2\bar{\theta}^{\perp} K(\Phi^I, \bar{\Phi}_I, \Psi_I, \bar{\Psi}^I), \quad (1.33)$$

where K is the Legendre transform of L ,

$$K(\Phi^I, \bar{\Phi}_I, \Psi_I, \bar{\Psi}^I) = L(\Phi^I, \bar{\Phi}_I, \Sigma^I, \bar{\Sigma}_I) - \Sigma^I \Psi_I - \bar{\Sigma}_I \bar{\Psi}^I. \quad (1.34)$$

⁶An equivalent form of this class of Lagrangian appeared originally in [7]. The physical significance of the u^{i+} -independent class was first noted in [21, 22].

⁷The original literature [20, 23] used Υ , $\check{\Upsilon}$ and F , which involve only the inhomogeneous coordinate ζ . When dealing with the superconformal transformation properties of arctic multiplets (see [24, 25] where these were first worked out), the u^{1+} factor becomes consequential. Although we will not be dealing explicitly with superconformal models, we find it useful to keep the u^{1+} .

In accord with the results of [2], this nonlinear sigma model must be hyperkähler. One of the three Kähler two-forms is manifest, coinciding with the usual Kähler two-form constructed from the Kähler potential K . The structure of the second supersymmetry of the model (1.33) establishes that the other two Kähler two-forms are given by $\omega = (d\Phi^I \wedge d\Psi_I)|$ and its complex conjugate [30, 31]. These three Kähler two-forms are related in the usual way to the three complex structures defining the hyperkähler geometry.

The set of steps described above involving the elimination of auxiliaries and the Legendre transformation can be succinctly described by the projective superspace equations of motion [21, 23]

$$\frac{\partial F^{++}}{\partial \Upsilon^{I+}} = i\Gamma_I^+ , \quad \frac{\partial F^{++}}{\partial \check{\Upsilon}_I^+} = -i\check{\Gamma}^{I+} , \quad (1.35)$$

where Γ^+ is required to be an arctic superfield, with the lowest component in its arctic expansion identified with Ψ . As noted in [32] (see also the recent discussion in [23]), the three Kähler two-forms of this system can be combined into a single section of a two-form valued $\mathcal{O}(2)$ bundle,

$$\Omega^{++} = u^{1+}u^{2+} \left(\frac{1}{\zeta} d\Phi^I \wedge d\Psi_I - i d\Phi^a \wedge d\bar{\Phi}_b \frac{\partial^2 K}{\partial \Phi^a \partial \bar{\Phi}_b} + \zeta d\bar{\Phi}_I \wedge d\bar{\Psi}^I \right) | , \\ \Phi^a = (\Phi^I, \Psi_I) , \quad \bar{\Phi}_a = (\bar{\Phi}_I, \bar{\Psi}^I) . \quad (1.36)$$

Using the equations of motion (1.35), this can be written in three equivalent ways [23]

$$\Omega^{++} = (d\Upsilon^{I+} \wedge d\Gamma_I^+) | = -i d\Upsilon^{I+} \wedge d\check{\Upsilon}_I^+ \frac{\partial^2 F^{++}}{\partial \Upsilon^{I+} \partial \check{\Upsilon}_I^+} \Big| = (d\check{\Upsilon}_I^+ \wedge d\check{\Gamma}^{I+}) | \quad (1.37)$$

in terms of the full arctic and antarctic expansions. One may interpret the on-shell Lagrangian F^{++} as a generating function for the canonical transformation connecting the “arctic coordinates” Υ^{I+} and Γ_I^+ to the “antarctic coordinates” $\check{\Upsilon}_I^+$ and $\check{\Gamma}^{I+}$, while preserving the symplectic two-form Ω^{++} [23]. These observations regarding F^{++} form the second ingredient to the construction we will present in this paper. For now, let us merely point out the striking similarity between (1.14) and (1.37).

1.3 Hamiltonian mechanics and canonical transformations

The main construction of our paper is to use a complexified harmonic superspace to map the harmonic description of a nonlinear sigma model to a projective description. To keep our presentation self-contained, we briefly review an alternative construction of the action principle in Hamiltonian mechanics that we will encounter shortly in a more complicated setting.

Recall that in the usual formulation of one-dimensional Hamiltonian mechanics,⁸ one introduces the action functional $S[q(t); t_0, t_1]$ defined by

$$S = \int_{t_0}^{t_1} dt L = \int_{t_0}^{t_1} dt (p\dot{q} - H) . \quad (1.38)$$

⁸The generalization to n dimensions is straightforwardly accomplished by replacing $q \rightarrow q^I$ and $p \rightarrow p_I$.

The initial and final times t_0 and t_1 are fixed, as are the trajectory endpoints $q_0 = q(t_0)$ and $q_1 = q(t_1)$. Taking the path $q(t)$ which extremizes the action, S becomes a function of q_0, q_1, t_0 , and t_1 , with $q(t)$ required to obey Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \quad (1.39)$$

Solutions to Hamilton's equations automatically preserve the canonical two-form

$$\omega = dq \wedge dp, \quad (1.40)$$

so that ω is independent of t . The partial derivatives of S with respect to q_0 and q_1 are given by

$$\frac{\partial S}{\partial q_1} = p_1, \quad \frac{\partial S}{\partial q_0} = -p_0, \quad (1.41)$$

which identifies the action S as the generating function (of the first type) for the canonical transformation associated with time evolution from t_0 to t_1 .

We can construct another generating function (of the second type) for this canonical transformation via a Legendre transformation:

$$F(q_0, p_1, t_0, t_1) := p_1 q_1 - S, \quad (1.42)$$

which obeys

$$\frac{\partial F}{\partial q_0} = p_0, \quad \frac{\partial F}{\partial p_1} = q_1. \quad (1.43)$$

The function F can equivalently be written

$$F = \frac{1}{2} p_0 q_0 + \frac{1}{2} p_1 q_1 + \int_{t_0}^{t_1} dt \left(H - \frac{1}{2} p \dot{q} + \frac{1}{2} q \dot{p} \right), \quad (1.44)$$

which places q and p on a more symmetric footing. This alternative action is just as fundamental as S . Let us now take (1.44) as the *definition* for F with $q(t)$ and $p(t)$ arbitrary paths constrained only by $q(t_0) = q_0$ and $p(t_1) = p_1$. Under arbitrary deformations of the paths that leave q_0 and p_1 invariant, F is extremized precisely for those solutions $q(t)$ and $p(t)$ which obey Hamilton's equations. This alternative action for Hamiltonian mechanics works even for the case of a vanishing Hamiltonian where the usual action principle fails. For such a trivial system, Hamilton's equations are solved by $q(t) = q_0 = q_1$ and $p(t) = p_0 = p_1$. While the usual action S vanishes in such a case, the symplectic action F is given by $F = q_0 p_1$ and yields the correct canonical transformation.

It is possible to draw an analogy between harmonic and projective descriptions of non-linear sigma models and the symplectic system described above. The harmonic equations of motion (1.13) are analogous to Hamilton's equations (1.39), as noted in [16]. The projective Lagrangian F^{++} is analogous to the action F , as both act (on-shell) as the generating function for a canonical transformation on their respective symplectic systems.

The remainder of this paper is devoted to fleshing out this analogy. In section 2, we show how to perform a certain contour deformation so that we can interpret the equations (1.13) in terms of a one-parameter Hamiltonian system. The actual construction is given in section 3, where we explain how it works with the trivial case of a free hypermultiplet and then test it with the cases of Taub-NUT and Eguchi-Hanson. We conclude with a brief discussion.

2 An alternative coordinate system and the complexified S^2

We begin with the harmonics in the form (1.17), with ζ describing the inhomogeneous coordinate on \mathbb{CP}^1 . Following [10], we introduce a real coordinate t given by⁹

$$t = \frac{\zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} . \quad (2.1)$$

The north pole lies at $t = 0$ and the south pole lies at $t = 1$. The harmonics may now be parametrized in terms of u^{1+} , ζ , and t :

$$\begin{aligned} u^{i+} &= (u^{1+}, u^{2+}) = u^{1+}(1, \zeta) , \\ u_i^- &= \frac{1}{u^{1+}}(1 - t, 0) + \frac{1}{u^{2+}}(0, t) = \frac{1}{u^{1+}} \left(1 - t, \frac{t}{\zeta} \right) . \end{aligned} \quad (2.2)$$

The S^2 is described by ζ and t ; the complex variable u^{1+} will drop out of all explicit calculations. Under smile conjugation,

$$\widetilde{u^{1+}} = \zeta u^{1+} , \quad \widetilde{\zeta} = -\frac{1}{\zeta} , \quad \widetilde{t} \rightarrow 1 - t . \quad (2.3)$$

The advantage of this coordinate system is that the operator D^{++} takes an especially simple form:

$$D^{++} = u^{1+} u^{2+} \frac{\partial}{\partial t} . \quad (2.4)$$

If we could interpret the real variable t as an independent coordinate (*i.e.* relax the constraint (2.1)), then the harmonic equations of motion (1.13) would resemble Hamilton's equations.

In this coordinate system, the harmonic measure becomes

$$\int du = \int_0^1 dt \oint_{C(t)} \frac{d\zeta}{2\pi i \zeta} \quad (2.5)$$

where $C(t)$ is a contour along the latitude of fixed $|\zeta|^2 = t/(1 - t)$. Because harmonic actions are required to be globally defined, the only harmonic integrals one encounters are always of the form $\int du F$ where $F(u)$ is a globally defined weight-zero function with the expansion

$$F(u) = F(\zeta, t) = f + f^{(ij)} u_i^+ u_j^- + \dots . \quad (2.6)$$

⁹Our conventions differ slightly from [10] by the replacement $t \rightarrow 1 - t$.

Since the harmonic integral selects out only the zero mode, $\int du F = f$, we are free to deform the ζ contour to be any latitude we wish and we maintain this property. (Equivalently, we treat $F(\zeta, t)$ as holomorphic in ζ with singularities only at $\zeta = 0$ and $\zeta = \infty$.) This means we can take

$$\int du F(u) = \int_0^1 dt \oint_{C(t)} \frac{d\zeta}{2\pi i \zeta} F(\zeta, t) \implies \oint_C \frac{d\zeta}{2\pi i \zeta} \int_0^1 dt F(\zeta, t) \quad (2.7)$$

where ζ is no longer constrained to obey (2.1) and C is now a t -independent contour. (A nearly identical approach was used in [10].) For definiteness, we will take C to be along the equator. For a function $F(\zeta, t)$ of the form (2.6), this integral again selects out the zero mode.

This approach of deforming the ζ contour can be interpreted as performing a certain complexification of the S^2 , a concept which has reappeared several times in the harmonic superspace literature.¹⁰ A complete discussion is beyond the scope of this paper, but we briefly discuss some details relevant to our construction in appendix A.

Now let us apply this alternative coordinate system to a harmonic superspace action. Given the action (1.5)

$$S = \frac{1}{16} \int d^4x \int du (D^-)^2 (\bar{D}^-)^2 \mathcal{L}^{++}, \quad (2.8)$$

we may rearrange the integrand and measure to the form

$$S = \frac{1}{16} \int d^4x \oint_C \frac{d\zeta}{2\pi i \zeta} \frac{1}{(u^{\perp+} u^{\bar{2}+})^2} \int_0^1 dt (D^{\perp})^2 (\bar{D}^{\bar{1}})^2 \mathcal{L}^{++}, \quad (2.9)$$

after discarding a total derivative. Now define the projective Lagrangian

$$\mathcal{L}^{++} = \frac{i}{u^{\perp+} u^{\bar{2}+}} \int_0^1 dt \mathcal{L}^{++}. \quad (2.10)$$

It is analytic by construction. After the complexification, \mathcal{L}^{++} is a function of the auxiliary coordinates $u^{\perp+}$, ζ , and t , and so \mathcal{L}^{++} is a function of only $u^{\perp+}$ and ζ . The corresponding action can be written

$$S = \frac{1}{2\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ \mathcal{L}^{++}, \quad (2.11)$$

which is exactly the form of a projective superspace action.¹¹

We would like to check if (2.10) generates a reasonable projective action for a simple case. Note that the expression (2.10) is singular in two ways. First, it involves the harmonic prefactor $1/u^{\perp+} u^{\bar{2}+}$ which diverges at the poles. Second, the complexification of the harmonics will lead well-defined functions on S^2 to develop singularities (as noted

¹⁰See *e.g.* [5, 33] for the relevance of complex harmonics for superconformal transformations, and [34] for their importance in describing quaternionic sigma models.

¹¹A similar procedure was used in [9] by replacing regular functions on S^2 with functions which were holomorphic everywhere except for a small region near the poles.

in [10]). The second issue is a more complicated one, so for the moment let's stick to an action involving multiplets which are always globally defined even in the projective setting. The simplest such multiplet is the tensor multiplet $G^{++} = G^{ij} u_i^+ u_j^+$, which is a globally defined $\mathcal{O}(2)$ multiplet in both the harmonic and projective descriptions. It possesses a unique superconformal action, the improved tensor multiplet action, which takes a quite different form depending on whether one uses a harmonic or a projective realization. In the projective setting, the Lagrangian can be written [6]

$$\mathcal{L}^{++} = -G^{++} \log(G^{++}/i u^{1+} u^{2+}) \quad (2.12)$$

where its resemblance to the $\mathcal{N} = 1$ analogue is striking. In the harmonic realization, the action is quite different [35] (see also [5]). One requires the introduction of an auxiliary isovector $c^{++} = c^{ij} u_i^+ u_j^+$, normalized so that $c^2 = \frac{1}{2} c^{ij} c_{ij} = 1$, in terms of which the Lagrangian can be written

$$\mathcal{L}^{++} = \left(\frac{\ell^{++}}{1 + \sqrt{1 + \ell^{++} c^{--}}} \right)^2, \quad \ell^{++} = G^{++} - c^{++}. \quad (2.13)$$

A global $SU(2)$ rotation allows the choice $c_{12} = i$ so that $c^{++} = 2i u^{1+} u^{2+}$. Now we complexify the harmonics (2.2) so that

$$c^{--} = -2i u_1^- u_2^- = \frac{2}{i u^{1+} u^{2+}} t(1-t). \quad (2.14)$$

Performing the t integral in (2.10) immediately gives

$$\mathcal{L}^{++} = G^{++} - c^{++} - G^{++} \log(G^{++}/c^{++}), \quad (2.15)$$

which agrees with (2.12) up to terms which do not contribute to the action. This verifies the reasonableness of the complexification procedure for a fairly nontrivial example. Now let us move on to a more complicated multiplet.

3 Deriving projective nonlinear sigma models and solutions from harmonic superspace

In this section, we first give our main result: the mapping of general nonlinear sigma models in harmonic superspace to nonlinear sigma models in projective superspace. For notational simplicity, we restrict to the case of a single hypermultiplet, but our results generalize straightforwardly to the case of n hypermultiplets. We also give the explicit on-shell relation for three example cases: the free hypermultiplet, Taub-NUT, and Eguchi-Hanson.

3.1 Nonlinear sigma models on the complexified S^2

Following [10], we begin by complexifying the functions q^+ and p^+ ,

$$q^+ \rightarrow Q^+ = Q^+(u^{1+}, \zeta, t), \quad p^+ \rightarrow P^+ = P^+(u^{1+}, \zeta, t), \quad (3.1)$$

where ζ is no longer constrained to obey (2.1).¹² The analytic constraints remain

$$D_\alpha^+ Q^+ = \bar{D}_{\dot{\alpha}}^+ Q^+ = 0, \quad D_\alpha^+ P^+ = \bar{D}_{\dot{\alpha}}^+ P^+ = 0, \quad (3.2)$$

and so Q^+ and P^+ are analytic superfields. We introduce obvious notation for the boundary values of Q^+ and P^+ at $t = 0$ and $t = 1$, respectively,

$$Q_0^+ := Q^+(u^{\perp+}, \zeta, 0), \quad Q_1^+ := Q^+(u^{\perp+}, \zeta, 1) \quad (3.3a)$$

$$P_0^+ := P^+(u^{\perp+}, \zeta, 0), \quad P_1^+ := P^+(u^{\perp+}, \zeta, 1). \quad (3.3b)$$

If q^+ and p^+ are globally defined functions on S^2 , then Q_0^+ and P_0^+ must be regular at $\zeta = 0$, and Q_1^+ and P_1^+ must be regular at $\zeta = \infty$. We identify

$$Q_0^+ \equiv \Upsilon^+, \quad P_0^+ \equiv \Gamma^+, \quad Q_1^+ \equiv -\check{\Gamma}^+, \quad P_1^+ \equiv \check{\Upsilon}^+ \quad (3.4)$$

where Υ^+ and Γ^+ are arctic multiplets and $\check{\Upsilon}^+$ and $\check{\Gamma}^+$ are antarctic. Note that the smile conjugation of harmonic superspace (2.3) includes a reflection in the real parameter t . The above observations are in accord with the results of [10].

We want to construct an action for Q^+ and P^+ so that they end up obeying the complexified version of the harmonic equations (1.13),

$$D^{++}Q^+ = \frac{\partial H^{+4}}{\partial P^+}, \quad D^{++}P^+ = -\frac{\partial H^{+4}}{\partial Q^+}. \quad (3.5)$$

Because t and ζ are now independent variables, these equations are simply Hamilton's equations for a symplectic system: the “position” Q^+ and “momentum” P^+ merely possess additional dependence on the coordinates ζ and $u^{\perp+}$.

To do this, it turns out that we need to relax the assumption that Q_1^+ is regular at $\zeta = \infty$ and P_0^+ is regular at $\zeta = 0$. In other words, we assume Γ^+ and $\check{\Gamma}^+$ to possess full Laurent expansions.¹³ However, we retain the property that $Q_0^+ \equiv \Upsilon^+$ is regular at $\zeta = 0$ and similarly, $P_1^+ \equiv \check{\Upsilon}^+$ remains regular at $\zeta = \infty$. (This is in accord with the analogous symplectic problem, where we fix q_0 and p_1 but do not fix the opposite endpoints.) Then, inspired by eq. (1.44), we postulate the following action on the complexified S^2 :

$$S = \int_{\mathcal{P}^2} du \int d^4x d^4\theta^+ \left(-\frac{1}{2} P^+ D^{++} Q^+ + \frac{1}{2} Q^+ D^{++} P^+ + H^{+4} \right) \\ + \frac{i}{4\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ Q_0^+ P_0^+ + \frac{i}{4\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ Q_1^+ P_1^+. \quad (3.6)$$

The first integral is an obvious generalization of the harmonic action and involves an integral over a region \mathcal{P}^2 of the complexified S^2 (see the discussion in appendix A):

$$\int_{\mathcal{P}^2} du := \oint_C \frac{d\zeta}{2\pi i \zeta} \int_0^1 dt. \quad (3.7)$$

¹²The $u^{\perp+}$ dependence of these expressions is trivial: $Q^+(u^{\perp+}, \zeta, t) = u^{\perp+} Q(\zeta, t)$ and similarly for P^+ .

¹³The convergence of these expansions is unclear to us. It seems reasonable that Γ^+ and $\check{\Gamma}^+$ should be taken initially to be “tropical” superfields in the language of projective superspace, with singularities allowed at both $\zeta = 0$ and $\zeta = \infty$.

We have taken the contour C to be along the equator with $|\zeta| = 1$. The second and third integrals resemble “surface terms” and involve the same contour C . In this form, the action bears a very close resemblance to the harmonic superspace action (1.12). In fact, if we required Q^+ and P^+ to be globally defined on the real S^2 , then $P_0^+ \equiv \Gamma^+$ would be arctic (and $Q_1^+ \equiv -\check{\Gamma}^+$ would be antarctic) and the “surface terms” would vanish, giving back the original harmonic action.

Now let us rewrite the action in a more suggestive form. Noting that

$$\frac{1}{2\pi} \oint_C u^+ du^+ = - \oint_C \frac{d\zeta}{2\pi i \zeta} i u^{1+} u^{2+} , \quad (3.8)$$

we can write

$$S = \frac{1}{2\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ F^{++} , \quad (3.9)$$

where the analytic function F^{++} is given by

$$F^{++} = \frac{i}{2} Q_0^+ P_0^+ + \frac{i}{2} Q_1^+ P_1^+ + \frac{i}{u^{1+} u^{2+}} \int_0^1 dt \left(H^{+4} - \frac{1}{2} P^+ D^{++} Q^+ + \frac{1}{2} Q^+ D^{++} P^+ \right) . \quad (3.10)$$

Note that F^{++} is holomorphic in ζ due to the complexification and satisfies all the requirements of a projective Lagrangian. It is also completely analogous to the symplectic action (1.44) discussed in the introduction.

Let us vary the Lagrangian. We find

$$\begin{aligned} \delta F^{++} &= i \delta Q_0^+ P_0^+ + i Q_1^+ \delta P_1^+ \\ &+ \frac{i}{u^{1+} u^{2+}} \int_0^1 dt \left\{ \delta Q^+ \left(D^{++} P^+ + \frac{\partial H^{+4}}{\partial Q^+} \right) - \delta P^+ \left(D^{++} Q^+ - \frac{\partial H^{+4}}{\partial Q^+} \right) \right\} . \end{aligned} \quad (3.11)$$

If we fix the endpoints $Q_0^+ \equiv \Upsilon^+$ and $P_1^+ \equiv \check{\Upsilon}^+$, we recover the complexified version of the harmonic equations (3.5). In principle, these may be solved in terms of Υ^+ and $\check{\Upsilon}^+$. Reinserting the solution into the Lagrangian, we find

$$F^{++} = F^{++}(\Upsilon^+, \check{\Upsilon}^+, u^+) . \quad (3.12)$$

This exactly yields the projective superspace action principle.

Now suppose we allow the endpoints to vary. We find that in order for the action to vanish, we must have

$$0 = \frac{1}{2\pi} \oint_C u^+ du^+ \int d^4x d^4\theta^+ \left(i \delta Q_0^+ P_0^+ + i Q_1^+ \delta P_1^+ \right) \quad (3.13)$$

where $\delta Q_0^+ \equiv \delta \Upsilon^+$ is arctic and $\delta P_1^+ \equiv \delta \check{\Upsilon}^+$ is antarctic. The solution is familiar from projective superspace: $P_0^+ \equiv \Gamma^+$ must be arctic and similarly $Q_1^+ \equiv -\check{\Gamma}^+$ must be antarctic. We also discover

$$\frac{\partial F^{++}}{\partial \Upsilon^+} = i \Gamma^+ , \quad \frac{\partial F^{++}}{\partial \check{\Upsilon}^+} = -i \check{\Gamma}^+ , \quad (3.14)$$

when F^{++} is written in terms of the endpoints Υ^+ and $\check{\Upsilon}^+$.

Remarkably, the same action (3.6) yields the harmonic superspace action when restricted to globally defined functions on the real S^2 and also the projective superspace action when put partly on-shell. Similarly, the same solutions, the trajectories Q^+ and P^+ , must reduce on the real S^2 to the harmonic solutions q^+ and p^+ and at their endpoints to the projective solutions Υ^+ and Γ^+ . Moreover, the harmonic and projective descriptions of the two-form Ω^{++} , eqs. (1.14) and (1.37), become transparently identified.

Let us see how this construction works in practice.

3.2 Examples

We will restrict our examples to on-shell multiplets (*i.e.* globally defined multiplets on the real S^2) because their solutions in harmonic superspace are available. Our first example is a very simple one: the free hypermultiplet. This was considered already in [9, 10], but we include it here for completeness. Then we address the more challenging cases of Taub-NUT and Eguchi-Hanson.

Free hypermultiplet

The free hypermultiplet corresponds to the harmonic action (1.12) with a vanishing Hamiltonian $H^{+4} = 0$. The equations of motion are easily solved by

$$q^+ = q^i u_i^+ , \quad p^+ = p^i u_i^+ . \quad (3.15)$$

Now we consider its complexification,

$$Q^+ = q^i u_i^+ , \quad P^+ = p^i u_i^+ . \quad (3.16)$$

Since these have no dependence on u_i^- , they are t -independent:

$$Q^+ = Q_0^+ = Q_1^+ = \Upsilon^+ = -\check{\Gamma}^+ , \quad P^+ = P_0^+ = P_1^+ = \check{\Upsilon}^+ = \Gamma^+ . \quad (3.17)$$

The projective Lagrangian F^{++} is easy to calculate:

$$F^{++} = \frac{i}{2} Q_0^+ P_0^+ + \frac{i}{2} Q_1^+ P_1^+ = i \Upsilon^+ \check{\Upsilon}^+ . \quad (3.18)$$

The solution clearly obeys the projective superspace equations of motion (1.35).

Taub-NUT

Next we consider the nontrivial case of Taub-NUT. The harmonic action possesses the Hamiltonian

$$H^{+4} = -\frac{\lambda}{2} (q^+ p^+)^2 , \quad (3.19)$$

where λ is a real parameter. The solution is known [13]:

$$\begin{aligned} q^+ &= f^+ \exp \left(-\frac{\lambda}{2} (f^+ \tilde{f}^- + f^- \tilde{f}^+) \right) , & p^+ &= \tilde{f}^+ \exp \left(+\frac{\lambda}{2} (f^+ \tilde{f}^- + f^- \tilde{f}^+) \right) , \\ f^\pm &:= f^i u_i^\pm , & \tilde{f}^\pm &:= \tilde{f}^i u_i^\pm , \end{aligned} \quad (3.20)$$

for some superfield f_i and its complex conjugate \bar{f}^i . Note that $q^+p^+ = f^+\tilde{f}^+$ is independent of u_i^- , and the Hamiltonian

$$H^{++} = -\frac{\lambda}{2}(f^+\tilde{f}^+)^2 \quad (3.21)$$

is also independent of u_i^- .

Now let us complexify. We take

$$Q^+ = f^+ \exp\left(-\frac{\lambda}{2}(f^+\tilde{f}^- + f^-\tilde{f}^+)\right), \quad P^+ = \tilde{f}^+ \exp\left(+\frac{\lambda}{2}(f^+\tilde{f}^- + f^-\tilde{f}^+)\right) \quad (3.22)$$

where u^{i+} and u_i^- are given by (2.2) but without imposing (2.1). We observe that $Q^+P^+ = Q_0^+P_0^+ = Q_1^+P_1^+$ is independent of u_i^- and therefore of t . The on-shell projective Lagrangian is given by

$$\begin{aligned} F^{++} &= \frac{i}{2}Q_0^+P_0^+ + \frac{i}{2}Q_1^+P_1^+ + \frac{i}{u^{1+}u^{2+}} \int_0^1 dt \left(H^{++} - \frac{1}{2}P^+D^{++}Q^+ + \frac{1}{2}Q^+D^{++}P^+ \right) \\ &= iQ_0^+P_0^+ + \frac{i}{u^{1+}u^{2+}} \int_0^1 dt \left(H^{++} - \frac{1}{2}P^+\frac{\partial H^{++}}{\partial P^+} - \frac{1}{2}Q^+\frac{\partial H^{++}}{\partial Q^+} \right) \\ &= iQ_0^+P_0^+ - \frac{i}{u^{1+}u^{2+}}H^{++}. \end{aligned} \quad (3.23)$$

We can write this in terms of f^+ and \tilde{f}^+ as

$$F^{++} = i f^+ \tilde{f}^+ - \frac{\lambda}{2i u^{1+}u^{2+}} (f^+ \tilde{f}^+)^2. \quad (3.24)$$

For shorthand, let us introduce the real analytic multiplet

$$U^{++} = i Q^+ P^+. \quad (3.25)$$

On-shell, we see that U^{++} is actually an $\mathcal{O}(2)$ multiplet,

$$U^{++} = i f^+ \tilde{f}^+ \implies D^{++}U^{++} = 0. \quad (3.26)$$

We also introduce the constant $c^{++} = i u^{1+}u^{2+}$ for convenience. Then we have

$$F^{++} = U^{++} + \frac{\lambda}{2} \frac{(U^{++})^2}{c^{++}}. \quad (3.27)$$

This is not useful quite yet since we require F^{++} to be written in terms of Υ^+ and $\check{\Upsilon}^+$. Observing that

$$\Upsilon^+ \equiv Q_0^+ = f^+ \exp\left(-\frac{\lambda}{2u^{1+}}(f^+\bar{f}^1 + f^1\bar{f}^+)\right), \quad (3.28a)$$

$$\check{\Upsilon}^+ \equiv P_1^+ = \tilde{f}^+ \exp\left(+\frac{\lambda}{2u^{2+}}(f^+\bar{f}^2 + f^2\bar{f}^+)\right), \quad (3.28b)$$

we find

$$i \Upsilon^+ \check{\Upsilon}^+ = i f^+ \tilde{f}^+ \exp\left(\lambda \frac{f^+ \tilde{f}^+}{u^{1+}u^{2+}}\right) = U^{++} \exp(\lambda U^{++}/c^{++}). \quad (3.29)$$

This relation can be inverted to solve for U^{++} in terms of $\Upsilon^+\check{\Upsilon}^+$:

$$U^{++} = \frac{c^{++}}{\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right), \quad (3.30)$$

where $W(z)$ is the Lambert W -function.¹⁴ Reinserting into the Lagrangian gives

$$F^{++} = \frac{c^{++}}{\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right) + \frac{c^{++}}{2\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right)^2. \quad (3.31)$$

Now let us check that this Lagrangian is indeed the Taub-NUT Lagrangian in projective superspace and that (3.28) gives the solution. The Taub-NUT action is expressed naturally as the sum of the free and improved tensor multiplet actions [32]

$$\mathcal{L}^{++} = G^{++} - G^{++} \log\left(G^{++}/i\Upsilon^+\check{\Upsilon}^+\right) - \frac{\lambda}{2} \frac{(G^{++})^2}{c^{++}}, \quad (3.32)$$

where G^{++} is an $\mathcal{O}(2)$ (or tensor) multiplet and $c^{++} = iu^{1+}u^{2+}$ is a real constant.¹⁵ The arctic multiplet Υ^+ is a pure gauge degree of freedom so long as G^{++} is an $\mathcal{O}(2)$ multiplet. We may derive this from the first-order Lagrangian

$$\mathcal{L}_{\text{F.O.}}^{++} = U^{++} - U^{++} \log\left(U^{++}/i\Upsilon^+\check{\Upsilon}^+\right) - \frac{\lambda}{2} \frac{(U^{++})^2}{c^{++}}, \quad (3.33)$$

where U^{++} is a real unconstrained analytic multiplet. Now the arctic multiplet's presence is consequential: applying the Υ^+ equation of motion leads to the condition that U^{++} is an $\mathcal{O}(2)$ multiplet. Instead, let us apply the U^{++} equation of motion. This leads to

$$i\Upsilon^+\check{\Upsilon}^+ = U^{++} \exp(\lambda U^{++}/c^{++}) \implies U^{++} = \frac{c^{++}}{\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right). \quad (3.34)$$

Reinserting this into the Lagrangian leads to the hypermultiplet action for Taub-NUT:

$$F^{++} = \frac{c^{++}}{\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right) + \frac{c^{++}}{2\lambda} W\left(i\lambda\Upsilon^+\check{\Upsilon}^+/c^{++}\right)^2. \quad (3.35)$$

This indeed matches (3.31). Now let's demonstrate that (3.28) gives the correct solution. Recall that Υ^+ must obey the equation of motion

$$\frac{\partial F^{++}}{\partial \Upsilon^+} = i\Gamma^+ \quad (3.36)$$

where Γ^+ is some arctic multiplet. This can be written

$$i\Gamma^+ = \frac{\partial \mathcal{L}_{\text{F.O.}}^{++}}{\partial U^{++}} \frac{\partial U^{++}}{\partial \Upsilon^+} = \left(1 + \lambda U^{++}/c^{++}\right) \frac{\partial U^{++}}{\partial \Upsilon^+} \quad (3.37)$$

where U^{++} satisfies (3.34). Imposing the solution (3.28), we see that (3.34) is solved by $U^{++} = i f^+ \tilde{f}^+$. Taking the Υ^+ derivative of (3.34), we find

$$i\check{\Upsilon}^+ = \left(1 + \lambda U^{++}/c^{++}\right) \exp(\lambda U^{++}/c^{++}) \frac{\partial U^{++}}{\partial \Upsilon^+}, \quad (3.38)$$

¹⁴The Lambert W -function $W(z)$ is defined as the solution $w = W(z)$ to the equation $z = we^w$.

¹⁵We have included the G^{++} term for convenience. It does not contribute to the action.

which implies that

$$\Gamma^+ = \exp(-\lambda U^{++}/c^{++}) \check{\Upsilon}^+ = \tilde{f}^+ \exp\left(\frac{\lambda}{2u^{\perp+}}(f^+ \bar{f}^{\perp} + f^{\perp} \tilde{f}^+)\right). \quad (3.39)$$

As required, this expression is arctic. As a check, we note that Γ^+ is indeed given by the $t = 0$ boundary of the complexified P^+ hypermultiplet: $\Gamma^+ = P^+(u^{\perp+}, \zeta, 0)$. Similarly, $\check{\Upsilon}^+$ is antartic and given by $\check{\Upsilon}^+ = -Q^+(u^{\perp+}, \zeta, 1)$. Thus (3.28) gives the solution for Taub-NUT in projective superspace.

It should be mentioned that although (3.28) is the solution, it is not in a useful form because the superfields f_i and \bar{f}^i are quite complicated. One should introduce the $\mathcal{N} = 1$ chiral superfields Φ and Ψ , which are found by taking the $\zeta = 0$ limits of Υ^+ and Γ^+ , respectively. They are

$$\Phi = -f_{\underline{1}} \exp\left(\frac{\lambda}{2}(f_{\underline{1}} \bar{f}^{\underline{1}} - f_{\underline{2}} \bar{f}^{\underline{2}})\right), \quad \Psi = \bar{f}^{\underline{2}} \exp\left(-\frac{\lambda}{2}(f_{\underline{1}} \bar{f}^{\underline{1}} - f_{\underline{2}} \bar{f}^{\underline{2}})\right). \quad (3.40)$$

In principle, one can solve for $f_{\underline{1}}$ and $f_{\underline{2}}$ in terms of Φ and Ψ and then reinsert the result into the expressions for Υ^+ and Γ^+ . One can even (in principle) construct the $\mathcal{N} = 1$ Lagrangian K indirectly using the explicit forms of Ω^{++} , (1.36) and (1.37), to extract the Kähler metric and then to integrate it.

Eguchi-Hanson

Our last example is Eguchi-Hanson. The harmonic action involves the Hamiltonian [36] (see also [37])

$$H^{+4} = \frac{2}{r^2}(g^{++})^2, \quad g^{++} := \frac{\ell^{++}}{1 + \sqrt{1 + 4\ell^{++}\xi^{--}/r^4}}, \quad \ell^{++} := -i q^+ p^+, \quad (3.41)$$

where $\xi^{--} = \xi^{ij} u_i^- u_j^-$ is a constant isotriplet normalized as $r^2 = \sqrt{2\xi^{ij}\xi_{ij}}$. We specialize to the case $\xi_{\underline{1}\underline{2}} = i r^2/2$ and $\xi_{\underline{1}\underline{1}} = \xi_{\underline{2}\underline{2}} = 0$. One solution is

$$q^+ = r \frac{f_2^+}{f_1^+} \left(u^{\underline{1}+} - \frac{2i}{r^2} u_{\underline{2}}^- g^{++}\right), \quad p^+ = r \frac{\tilde{f}_2^+}{f_1^+} \left(u^{\underline{2}+} - \frac{2i}{r^2} u_{\underline{1}}^- g^{++}\right), \quad (3.42a)$$

$$f_a^{\pm} := f_a^i u_i^{\pm}, \quad \tilde{f}_a^{\pm} := \bar{f}_a^i u_i^{\pm}, \quad a = 1, 2 \quad (3.42b)$$

where f_a^+ is constrained by

$$\xi^{++} = i f_1^+ \tilde{f}_1^+ + i f_2^+ \tilde{f}_2^+, \quad (3.43)$$

and g^{++} is a function of ℓ^{++} , which is given by

$$\ell^{++} = -i q^+ p^+ = -i f_2^+ \tilde{f}_2^+. \quad (3.44)$$

As with the Taub-NUT solution, the product of q^+ and p^+ is simple and t -independent, $q^+ p^+ = f_2^+ \tilde{f}_2^+$.

The constrained complex coordinates f_a^i correspond to those originally discussed in [3, 38]. These models were shown to derive from a harmonic action involving two minimally

coupled hypermultiplets and an auxiliary vector multiplet in [37] (based on the construction given in $\mathcal{N} = 1$ language [39, 40]), which can be understood as the universal description of the Eguchi-Hanson model in harmonic superspace. The version (3.41) we have given here corresponds to dualizing one of the hypermultiplets in [37] into an (ω, L^{++}) system and then applying the vector multiplet equation of motion, as discussed in [5].

The solution (3.42) is valid only in a certain coordinate patch where $|f_{12}|^2 < |f_{11}|^2$. There is another solution found by exchanging $f_1^+ \leftrightarrow f_2^+$, which is valid for $|f_{22}|^2 < |f_{21}|^2$. These two patches cover the entire region defined by (3.43). The reason two solutions are needed is apparent by noting that q^+ must be well-defined over the real S^2 , but the first solution has an apparent singularity when ζ is chosen so that f_1^+ vanishes, that is, $\zeta = -f_{11}/f_{12}$. In the first coordinate patch, this occurs for $|\zeta| > 1$, and one can check that for this range, the apparent singularity in q^+ is actually resolved: as $\zeta \rightarrow -f_{11}/f_{12}$, one finds $2i u_2^- g^{++} \rightarrow u^{1+} r^2$ and q^+ approaches a finite value. Similarly, p^+ has an apparent singularity at the antipodal point, which is resolved in the same fashion.

Now let us derive the Eguchi-Hanson model in projective superspace. The on-shell projective Lagrangian is (after complexifying)

$$\begin{aligned} F^{++} &= \frac{i}{2} Q_0^+ P_0^+ + \frac{i}{2} Q_1^+ P_1^+ + \frac{i}{u^{1+} u^{2+}} \int_0^1 dt \left(H^{+4} - \frac{1}{2} P^+ D^{++} Q^+ + \frac{1}{2} Q^+ D^{++} P^+ \right) \\ &= -i f_2^+ \tilde{f}_2^+ + \frac{i}{u^{1+} u^{2+}} \int_0^1 dt \left(H^{+4} - \frac{1}{2} P^+ \frac{\partial H^{+4}}{\partial P^+} - \frac{1}{2} Q^+ \frac{\partial H^{+4}}{\partial Q^+} \right) \\ &= -i f_2^+ \tilde{f}_2^+ + \frac{i}{u^{1+} u^{2+}} \int_0^1 dt \left(\frac{2}{r^2} \frac{\xi^{++}}{i f_1^+ \tilde{f}_1^+} (g^{++})^2 + \frac{2}{r^2} D^{++} (\xi^{+-} g^{++}) \frac{f_2^+ \tilde{f}_2^+}{f_1^+ \tilde{f}_1^+} \right) \end{aligned}$$

There are two t -integrals to do. The first is nearly identical to the integral we performed which led to (2.15). It yields

$$\frac{\xi^{++}}{i f_1^+ \tilde{f}_1^+} (\ell^{++} - (\ell^{++} + \xi^{++}) \ln(1 + \ell^{++}/\xi^{++})) \quad (3.45)$$

The second is a total derivative, which simplifies its evaluation:

$$-\ell^{++} \frac{f_2^+ \tilde{f}_2^+}{f_1^+ \tilde{f}_1^+} . \quad (3.46)$$

Combining all these terms and making use of the constraint (3.43), we find

$$F^{++} = -\xi^{++} \ln(1 + \ell^{++}/\xi^{++}) . \quad (3.47)$$

Now we must identify the arctic multiplet Υ^+ and its conjugate $\check{\Upsilon}^+$. They are given by

$$\Upsilon^+ = Q_0^+ = r u^{1+} \frac{f_2^+}{f_1^+} , \quad \check{\Upsilon}^+ = P_1^+ = r u^{2+} \frac{\tilde{f}_2^+}{\tilde{f}_1^+} . \quad (3.48)$$

One can check that

$$1 + \ell^{++}/\xi^{++} = \frac{1}{1 + i \Upsilon^+ \check{\Upsilon}^+ / \xi^{++}} \quad (3.49)$$

which yields

$$F^{++} = \xi^{++} \ln \left(1 + i \Upsilon^+ \check{\Upsilon}^+ / \xi^{++} \right) . \quad (3.50)$$

This is indeed the projective action for Eguchi-Hanson [21]. Now let's check that it obeys the equations of motion. We observe that

$$\Gamma^+ := P_0^+ = r \frac{\tilde{f}_2^+}{\tilde{f}_1^+} \left(u^{2+} - \frac{i}{r^2 u^{1+}} \ell^{++} \right) = \frac{\check{\Upsilon}^+}{1 + i \Upsilon^+ \check{\Upsilon}^+ / \xi^{++}} = -i \frac{\partial F^{++}}{\partial \Upsilon^+} \quad (3.51)$$

is arctic. Similarly, $\check{\Gamma}^+ := -Q_1^+$ is antarctic and obeys (1.35).

Finally, we note that the explicit solution for Υ^+ (3.48) can be shown to match that given originally in [21]. One introduces the variables Φ and Σ defined as the first two terms in the ζ expansion of Υ^+ :

$$\Upsilon^+ = u^{1+} (\Phi + \zeta \Sigma + \dots) . \quad (3.52)$$

One finds Φ and Σ are given by

$$\Phi = r \frac{f_{21}}{f_{11}} , \quad \Sigma = r \frac{f_{22}}{f_{11}} (1 + \Phi \bar{\Phi} / r^2) . \quad (3.53)$$

In terms of these variables, the solution for Υ^+ becomes

$$\Upsilon^+ = u^{1+} \frac{\Phi + \zeta \Sigma / (1 + \Phi \bar{\Phi} / r^2)}{1 - \zeta \bar{\Phi} \Sigma / (r^2 + \Phi \bar{\Phi})} \quad (3.54)$$

in complete agreement with [21].

4 Discussion

For brevity's sake, we have given only the construction relating harmonic and projective descriptions of nonlinear sigma models and two nontrivial examples here. Many other nice examples, including several broad classes, remain to be understood. One particular class of interest is that of superconformal nonlinear sigma models: these are known in both harmonic [33] and in projective superspace [24, 25]. Another class is the set of nonlinear sigma models on tangent bundles of Hermitian symmetric spaces, which have been fully analyzed from the projective approach [21, 22, 26–29].

An important consequence of this construction is that it points toward a unifying framework for the harmonic and projective descriptions of hyperkähler target spaces. It was shown in [14] that the harmonic description of hyperkähler geometry was universal: all hyperkähler metrics come from some harmonic nonlinear sigma model (at least locally). It was similarly argued in [23] that the same holds for projective descriptions. It would be quite interesting to see if one could “complexify” the geometric approach taken in [14] to construct a projective analogue.

One interesting possibility that we have avoided addressing is whether it might be possible to reconstruct harmonic actions and solutions directly from projective solutions.

On the one hand, this seems impossible since information has been lost in moving from the harmonic to the projective formulation. Indeed, this is the dominant point of view [9, 10]. However, the analogy to Hamiltonian mechanics that we discussed in the introduction suggests an alternative. If we view the symplectic action F as a function of not just the endpoint positions but also the endpoint *times*, then the Hamiltonian can be reconstructed by varying either of the endpoints,

$$\frac{\partial F}{\partial t_1} = H(q_1, p_1, t_1) , \quad \frac{\partial F}{\partial t_0} = -H(q_0, p_0, t_0) . \quad (4.1)$$

Certainly we cannot vary the “times” in the complexified S^2 since the boundary values of t correspond to something meaningful – the two poles. However, it might be possible to vary the location of the poles themselves. Presumably, a single harmonic action might then correspond to a family of projective actions. This conjecture probably exceeds the elasticity of the analogy, but it may be interesting to investigate further.

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A The complexified S^2

The two-sphere S^2 is defined as the set of points $X^I \in \mathbb{R}^3$ obeying $\sum_I (X^I)^2 = 1$. It can be related to the space \mathbb{CP}^1 , with homogeneous coordinate v^i , via the identification $X^I = \bar{v}_j (\sigma^I)^{j_k} v^k / (v, \bar{v})$ where σ^I are the usual Pauli matrices. This defines v^i up to the equivalence relation $v^i \sim cv^i$ for complex $c \neq 0$.

This space is straightforwardly complexified¹⁶ to the standard affine quadric Q^2 , defined as (see *e.g.* [41])

$$Q^2 = \left\{ Z^I \in \mathbb{C}^3 : \sum_I (Z^I)^2 = 1 \right\} . \quad (A.1)$$

One can check that $Q^2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ by identifying

$$Z^I = \frac{1}{(v, \bar{w})} \bar{w}_j (\sigma^I)^{j_k} v^k , \quad (v, \bar{w}) := v^k \bar{w}_k . \quad (A.2)$$

This defines v^i and \bar{w}_j up to the identifications

$$v^i \sim cv^i , \quad \bar{w}_j \sim \bar{d}\bar{w}_j , \quad c, \bar{d} \in \mathbb{C} - \{0\} . \quad (A.3)$$

Q^2 can be identified with $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the region $(v, \bar{w}) = 0$ removed.

This complexified two-sphere obviously has a real S^2 as a compact subspace with $Z^I = X^I$, or $v^i \propto w^i$:

$$S^2 = \left\{ (v^i, \bar{w}_i) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : v^i \propto w^i \right\} . \quad (A.4)$$

¹⁶The author would like to thank Sergei Kuzenko for educating him about the complexified S^2 .

The standard integral on the S^2 is (1.18), which we can interpret as

$$\int_{S^2} du = \frac{1}{2\pi i} \int_{S^2} \frac{v^i dv_i \wedge \bar{w}_j d\bar{w}^j}{(v, \bar{w})^2} , \quad (\text{A.5})$$

for $S^2 \subset Q^2$ defined by (A.4).

Now suppose we have a scalar function F defined on S^2 . It possesses a convergent harmonic expansion (2.6). F may be analytically continued to a bi-holomorphic function on Q^2

$$F = f + f^{(ij)} \frac{v_i \bar{w}_j}{(v, \bar{w})} + \dots \quad (\text{A.6})$$

which in general only converges in the vicinity of $v \propto w$. Along S^2 , we know that

$$\int_{S^2} du F = f . \quad (\text{A.7})$$

We would like to extend this result to a different surface on Q^2 . Let us introduce the complex coordinates ζ and t given by

$$v^i = v^{\perp}(1, \zeta) , \quad \bar{w}_i = \frac{\bar{w}_1}{1-t} (1-t, t/\zeta) . \quad (\text{A.8})$$

In terms of these coordinates,

$$\int_{S^2} du F = \int_{S^2} \frac{d\zeta \wedge dt}{2\pi i \zeta} F \quad (\text{A.9})$$

where S^2 can now be identified as the region

$$S^2 = \left\{ (\zeta, t) : 0 \leq t \leq 1 , |\zeta|^2 = \frac{t}{1-t} \right\} . \quad (\text{A.10})$$

Now we keep the domain of t fixed but deform the ζ contour to $|\zeta| = 1$. We must assume that the expansion (A.6) remains valid after the deformation. Let \mathcal{P}^2 be the corresponding region with $|\zeta| = 1$. To prove that (A.7) holds, we must prove that the harmonic result

$$\int_{S^2} du u^{i_1+} \dots u^{i_n+} u_{j_1}^- \dots u_{j_n}^- = \frac{n!}{(n+1)!} \delta_{(j_1}^{(i_1} \dots \delta_{j_n)}^{i_n)} \quad (\text{A.11})$$

holds when we replace S^2 with \mathcal{P}^2 , for which

$$u^{i+} = u^{\perp+}(1, \zeta) , \quad u_j^- = \frac{1}{u^{\perp+}} (1-t, t/\zeta) . \quad (\text{A.12})$$

This is quite straightforward. We first observe that the ζ contour integral is non-vanishing only for ζ -independent harmonic products. This implies that we must have the same number of $u^{\perp+}$ and u_{\perp}^- factors, which implies the Kronecker delta structure. So we are reduced to checking for the non-vanishing case of p $u^{\perp+}$'s and $n-p$ $u^{\perp+}$'s:

$$\int_{\mathcal{P}^2} du (u^{\perp+} u_{\perp}^-)^p (u^{\perp+} u_{\perp}^-)^{n-p} = \frac{p! (n-p)!}{(n+1)!} \quad (\text{A.13})$$

Performing the ζ integral, we find

$$\int_0^1 dt (1-t)^p t^{n-p} = \frac{p! (n-p)!}{(n+1)!} \quad (\text{A.14})$$

which does indeed hold. So we are justified in performing the continuation of S^2 to \mathcal{P}^2 (dubbed a “Wick rotation” in [10]) provided that the integrand still possesses a convergent expansion.

Finally, we should comment briefly about what the surface \mathcal{P}^2 looks like as a subspace of $Q^2 \subset \mathbb{C}P^1 \times \mathbb{C}P^1 = S^2 \times S^2$. It is not hard to see that

$$\mathcal{P}^2 = \left\{ (\theta_1, \phi_1, \theta_2, \phi_2) \in S^2 \times S^2 : \phi_1 = \phi_2, \theta_1 = \pi/2 \right\}. \quad (\text{A.15})$$

In other words, \mathcal{P}^2 arises when we identify the azimuthal angles on the two S^2 ’s and restrict to the equator of the first S^2 . In particular, \mathcal{P}^2 is a compact surface.

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